

AFFINE CONNECTIONS, DUALITY AND DIVERGENCES FOR A VON NEUMANN ALGEBRA.

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Abstract. On the predual of a von Neumann algebra, we define a differentiable manifold structure and affine connections by embeddings into non-commutative L_p -spaces. Using the geometry of uniformly convex Banach spaces and duality of the L_p and L_q spaces for $1/p + 1/q = 1$, we show that we can introduce the α -divergence, for $\alpha \in (-1, 1)$, in a similar manner as Amari in the classical case. If restricted to the positive cone, the α -divergence belongs to the class of quasi-entropies, defined by Petz.

1. INTRODUCTION

The classical information geometry deals with the differential geometric aspects of families of probability densities with respect to a given measure μ . The theory, developed in [1, 5], has been already extended to the nonparametric case, where the manifold is modelled on some infinite dimensional Banach space, see [20, 7].

One of the important results of Amari's classical (finite dimensional) information geometry [1, 2] deals with the structure of Riemannian manifolds with a pair of flat affine connections, dual with respect to the metric. For such manifolds, there is a pair (θ, η) of dual affine coordinate systems, related by Legendre transformations

$$\theta_i = \frac{\partial}{\partial \eta_i} \varphi(\eta) \quad \eta_i = \frac{\partial}{\partial \theta_i} \psi(\theta),$$

where ψ, φ are potential functions. A quasi-distance, called the divergence, is then defined by

$$D(\theta_1, \theta_2) = \psi(\theta_1) + \varphi(\eta_2) - \sum_i \theta_{1i} \eta_{2i}$$

For manifolds of probability density functions, flat with respect to the $\pm\alpha$ -connections, the corresponding α -divergence belongs to the class

of Cziszár's f -divergences

$$S_f(p, q) = \int f\left(\frac{q}{p}\right) dp$$

where f is a convex function. The f -divergences were generalized to von Neumann algebras by Petz in [19] by means of the relative modular operator of normal positive functionals on M :

$$S_g(\phi, \psi) = (g(\Delta_{\phi, \psi})\xi_\psi, \xi_\psi)$$

where ξ_ψ is the vector representative of ψ . On the other hand, Amari's construction of the α -divergence, starting from a pair of dual flat connections, was extended to the manifold of faithful positive linear functionals on a matrix algebra $\mathcal{M}_n(\mathbb{C})$, [13, 10]. The aim of the present paper is to show that there is such a construction for a general von Neumann algebra.

For $\alpha \in (-1, 1)$, the α -connections can be defined using α -embeddings into non-commutative L_p -spaces, $p = \frac{2}{1-\alpha}$. In this case, the α and $-\alpha$ -connections are defined on different vector bundles and their duality corresponds to the Banach space duality of L_p and L_q , $1/p + 1/q = 1$, therefore this duality does not require a Riemannian metric. This was shown by Gibilisco and Isola in [8] (see also [7] for the classical case). Here, the α -embeddings were used to define the α -connections on manifolds of faithful density operators of a semifinite von Neumann algebra. The manifold structure, however, was not specified here, although some definitions of such a structure already appeared, see [11, 21, 22].

Another possibility is to use the α -embedding to introduce the manifold structure. Here the problem is, that the range of the α -embedding is in the positive cone of the L_p -space which, even in the classical case, can have empty interior. This problem was avoided in [14], in defining the α -embedding on the whole predual M_* and not just on the positive cone.

The α -connections are defined as the trivial connections in $L_p(M, \phi)$ and the $\pm\alpha$ -duality is just the Banach space duality. The $\pm\alpha$ -embeddings define a pair of dual coordinates on M_* . Using the fact that the L_p spaces with $p \in (1, \infty)$ are uniformly convex, it was shown that the dual coordinates are related by potential functionals, just as in Amari's theory. From this, we can define a divergence functional on $L_p(M, \phi)$.

Via the α -embedding, the divergence in $L_p(M, \phi)$ induces a functional on $M_* \times M_*$, which is called the α -divergence. We will show that if restricted to the positive cone, the α -divergence is exactly the Petz

quasi-entropy S_{g_α} , with

$$g_\alpha(t) := \frac{2}{1-\alpha} + \frac{2}{1+\alpha}t - \frac{4}{1-\alpha^2}t^{\frac{1+\alpha}{2}}.$$

We will further investigate the properties of the divergence in $L_p(M, \phi)$, especially the projection theorems. These imply some existence and uniqueness results for the α -projections, which generalize the projection theorems in [1].

2. UNIFORMLY CONVEX BANACH SPACES.

We recall some facts about convexity and smoothness in Banach spaces, see [15].

Let X be a Banach space and let X^* be the dual of X . Then for $u \in X^*$ we denote $\langle x, u \rangle = u(x)$. Let K be a closed convex subset in X with nonempty interior, in particular, let K_d be closed ball with radius d . Let S be the boundary of K .

A supporting hyperplane of K is a real hyperplane $x + H$, containing at least one point of K and such that K lies in one of the two closed half-spaces determined by $x + H$. There is at least one supporting hyperplane through every boundary point of K . A boundary point $x_0 \in S$ is called a point of smoothness if exactly one closed supporting hyperplane passes through x_0 , called a tangent hyperplane. We say that K is smooth if every boundary point is a point of smoothness. The space X is called smooth if K_1 is smooth.

A normed space is smooth if and only if the norm is weakly differentiable at each point except the origin. The weak derivative of the norm at x_0 in the direction y is given by $\Re \langle y, v_{x_0/\|x_0\|} \rangle$, where $v_{x_0/\|x_0\|}$ is the unique point in the unit sphere of X^* , satisfying $\langle x_0, v_{x_0/\|x_0\|} \rangle = \|x_0\|$ and \Re denotes the real part. The tangent hyperplane to the sphere $S_{\|x_0\|}$ at x_0 is $x_0 + H$, with

$$H = \{x \in X, \Re \langle x, v_{x_0/\|x_0\|} \rangle = 0\}$$

The set K is said to be strictly convex if every boundary point of K is an extreme point, equivalently, the boundary of K contains no line segment. In this case, each supporting hyperplane meets K in exactly one point.

A reflexive Banach space is smooth if and only if its dual X^* is strictly convex, that is, the unit ball in X^* is strictly convex.

The space X and its closed unit ball, are said to be uniformly convex if for each ϵ , $0 < \epsilon \leq 2$ there is a $\delta(\epsilon) > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ always implies that $\|\frac{1}{2}(x + y)\| \leq 1 - \delta(\epsilon)$. The function

$\delta(\epsilon)$ is called the module of convexity. Every uniformly convex space is strictly convex and reflexive.

There is also a stronger notion of smoothness, dual to uniform convexity. The space X , and its norm, are said to be uniformly smooth if for each $\epsilon > 0$ there is an $\eta(\epsilon) > 0$, such that $\|x\| \geq 1$, $\|y\| \geq 1$ and $\|x - y\| \leq \eta(\epsilon)$ always implies $\|x + y\| \geq \|x\| + \|y\| - \epsilon\|x - y\|$.

A normed space X is uniformly smooth if and only if its norm is uniformly strongly differentiable. In particular, every uniformly smooth normed space is smooth. A Banach space X is uniformly convex (uniformly smooth) if and only if X^* is uniformly smooth (uniformly convex).

We will also need the following two results by Cudia [6].

Theorem 2.1. *Let S resp. S' be the unit sphere in X resp. X^* . The norm is (uniformly) strongly differentiable in S if and only if the map $v : x \mapsto v_x$ is single valued and (uniformly) continuous from the norm topology on S to the norm topology on S' .*

Let us now define the map $F : X \rightarrow X^*$ by

$$F(x) = \begin{cases} \|x\|v_{x/\|x\|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Theorem 2.2. *Let the Banach space X be uniformly convex and let the norm be strongly differentiable. Then F is a homeomorphism of X onto X^* (in the norm topologies).*

3. NON-COMMUTATIVE L_p -SPACES.

Let M be a von Neumann algebra and let ϕ be a faithful normal semifinite weight. We denote N_ϕ the set of $y \in M$ satisfying $\phi(y^*y) < \infty$ and M_0 the set of all elements in $N_\phi \cap N_\phi^*$, entire analytic with respect to the modular automorphism σ_t^ϕ associated with ϕ . We also denote the GNS map by $N_\phi \ni y \mapsto \eta_\phi(y) \in H_\phi$.

Let $1 \leq p \leq \infty$ and let $L_p(M, \phi)$ be the non-commutative L_p space with respect to ϕ , as defined by Araki and Masuda in [4, 17]. The elements of $L_p(M, \phi)$ are closed operators acting on the Hilbert space H_ϕ , satisfying

$$T J_\phi \sigma_{-i/p}^\phi(y) J_\phi \supset J_\phi y J_\phi T,$$

for all $y \in M_0$, such that the L_p -norm

$$\|T\|_{p,\phi} = \left\{ \sup_{x \in M_0, \|x\| \leq 1} \| |T|^{p/2} \eta_\phi(x) \| \right\}^{2/p}$$

is finite. Then $L_p(M, \phi)$ with the L_p -norm is a Banach space. Let $1 < p < \infty$, then $L_p(M, \phi)$ is uniformly convex and uniformly strongly

differentiable. The dual space $L_p^*(M, \phi)$ is $L_q(M, \phi)$, with $1/p + 1/q = 1$, where the duality is given by

$$(1) \quad \langle T, T' \rangle_\phi = \lim_{y \rightarrow 1} (T\eta_\phi(y), T'\eta_\phi(y))$$

where $T \in L_p(M, \phi)$, $T' \in L_q(M, \phi)$. The limit is taken in the *-strong topology with restriction $y \in M_0$, $\|y\| \leq 1$.

Each $T \in L_p(M, \phi)$, $1 \leq p < \infty$, has a unique polar decomposition of the form

$$T = u\Delta_{\psi, \phi}^{1/p}$$

where $\psi \in M_*^+$, $u \in M$ is a partial isometry, such that the support projection $s(\phi) = u^*u$ and $\Delta_{\psi, \phi}$ is the relative modular operator, see Appendix C in [4] for definition and basic properties. On the other hand, each operator of this form is in $L_p(M, \phi)$. The positive cone $L_p^+(M, \phi)$ is the set of positive operators in $L_p(M, \phi)$ and we have

$$L_p^+(M, \phi) = \{\Delta_{\psi, \phi}^{1/p}, \psi \in M_*^+\}$$

The identity

$$(2) \quad \varphi(au) = \langle u\Delta_{\varphi, \phi}, a^* \rangle_\phi$$

for $a \in M$ gives an isometric isomorphism of M_* and $L_1(M, \phi)$. Similarly, $L_2(M, \phi)$ is isomorphic to H_ϕ by

$$u\Delta_{\varphi, \phi}^{1/2} \mapsto u\xi_\varphi,$$

where ξ_φ is the vector representative of φ in the neutral positive cone in H_ϕ .

If $\tilde{\phi}$ is a different n.s.f. weight, then there is an isometric isomorphism $\tau_p(\tilde{\phi}, \phi) : L_p(M, \phi) \rightarrow L_p(M, \tilde{\phi})$ and

$$(3) \quad \langle T, T' \rangle_\phi = \langle \tau_p(\tilde{\phi}, \phi)T, \tau_q(\tilde{\phi}, \phi)T' \rangle_{\tilde{\phi}}$$

holds for all $T \in L_p(M, \phi)$ and $T' \in L_q(M, \phi)$.

A bilinear form on $L_p(M, \phi) \times L_q(M, \phi)$ is defined by

$$[T, T']_\phi = \langle T, T'^* \rangle_\phi, \quad T \in L_p(M, \phi), \quad T' \in L_q(M, \phi)$$

If $T_k \in L_{p_k}(M, \phi)$, $\sum_k 1/p_k = 1/r$, then the product $T = T_1 \dots T_n$ is well defined as an element of $L_r(M, \phi)$ and

$$\|T\|_r \leq \|T_1\|_{p_1} \dots \|T_n\|_{p_n}$$

If $r = 1$, then

$$(4) \quad \begin{aligned} [T_1 \dots T_n]_\phi &:= [T, 1]_\phi = [T_1 \dots T_k, T_{k+1} \dots T_n]_\phi = \\ &= [T_{k+1} \dots T_n T_1 \dots T_k]_\phi \end{aligned}$$

for each $1 \leq k \leq n-1$ and

$$(5) \quad |[T_1 \dots T_n]_\phi| \leq \|T_1\|_{p_1} \dots \|T_n\|_{p_n}$$

4. THE α -EMBEDDINGS AND AFFINE CONNECTIONS

Let M be a von Neumann algebra and let ϕ be a faithful normal semifinite weight.

For $1 < \alpha < 1$, we define the non-commutative α -embedding by

$$\begin{aligned} \ell_\alpha^\phi : M_* &\rightarrow L_p(M, \phi), \quad p = \frac{2}{1-\alpha} \\ \omega &\mapsto pu\Delta_{\varphi, \phi}^{1/p} \end{aligned}$$

where $\omega(a) = \varphi(au)$, $a \in M$ is the polar decomposition of ω . It is clear from uniqueness of the polar decompositions that ℓ_α^ϕ is bijective. Moreover, it maps the hermitian (that is, $\omega(a^*) = \overline{\omega(a)}$) elements in M_* onto the real Banach space $L_p^h(M, \phi)$ of self-adjoint operators in the L_p -space and M_*^+ onto the positive cone $L_p^+(M, \phi)$.

If ψ is a different f.n.s. weight, then the space $L_p(M, \psi)$ is identified with $L_p(M, \phi)$ by the isometric isomorphism $\tau_p(\psi, \phi)$. The corresponding α -embeddings are related by

$$\ell_\alpha^\psi = \tau_p(\psi, \phi)\ell_\alpha^\phi$$

We denote by \mathcal{M}_α the set M_* with the manifold structure induced from ℓ_α^ϕ . Due to the above isomorphism, the manifold structure does not depend from the choice of ϕ . For $\omega \in M_*$, $\ell_\alpha^\phi(\omega) \in L_p(M, \phi)$ will be called the α -coordinate of ω . The $-\alpha$ -coordinate is an element of the dual space $L_q(M, \phi)$, $1/p + 1/q = 1$. Moreover, for $\omega_1, \omega_2 \in M_*$ and a n.s.f. weight ψ , we have by (3)

$$\begin{aligned} (6) \quad \langle \ell_\alpha^\psi(\omega_1), \ell_{-\alpha}^\psi(\omega_2) \rangle_\psi &= \langle \tau_p(\psi, \phi)\ell_\alpha^\phi(\omega_1), \tau_q(\psi, \phi)\ell_{-\alpha}^\phi(\omega_2) \rangle_\psi = \\ &= \langle \ell_\alpha^\phi(\omega_1), \ell_{-\alpha}^\phi(\omega_2) \rangle_\phi \end{aligned}$$

In the sequel, we will just write ℓ_α instead of ℓ_α^ϕ . We will say that $\ell_\alpha(\omega)$ and $\ell_{-\alpha}(\omega) \in L_q(M, \phi)$ are dual coordinates of $\omega \in M_*$.

The trivial connection in $L_p(M, \phi)$ induces a globally flat affine connection on the tangent bundle $T\mathcal{M}_\alpha$, called the α -connection. Let us recall that there is a one-to-one correspondence between affine connections and parallel transports on $T\mathcal{M}_\alpha$. If the connection is globally flat, the parallel transport is given by a family of isomorphisms $U_{x,y} : T_x(\mathcal{M}_\alpha) \rightarrow T_y(\mathcal{M}_\alpha)$, $x, y \in \mathcal{M}_\alpha$, satisfying

- (i) $U_{x,x} = Id$,
- (ii) $U_{y,z}U_{x,y} = U_{x,z}$

In our case, the tangent space $T_x(\mathcal{M}_\alpha)$ can be identified with $L_p(M, \phi)$ and the map $U_{x,y}$ is the identity map for all $x, y \in \mathcal{M}_\alpha$. We define the dual connection as in [7], that is, a linear connection on the cotangent bundle $T^*\mathcal{M}_\alpha$, such that the corresponding parallel transport U^* satisfies

$$\langle v, U_{x,y}^*(w) \rangle_\phi = \langle U_{y,x}(v), w \rangle_\phi = \langle v, w \rangle_\phi$$

for $w \in (T_x(\mathcal{M}_\alpha))^* \equiv L_q(M, \phi)$ and $v \in T_y(\mathcal{M}_\alpha)$. Obviously, U^* is the trivial parallel transport in $L_q(M, \phi)$, hence the dual of the α -connection is the $-\alpha$ -connection.

5. DUALITY.

Let $\omega \in M_*$. We will show how ω is related to its dual coordinates.

Proposition 5.1. *Let $\omega \in M_*$, $\omega(a) = \psi(au)$ be the polar decomposition and let $\psi_u(a) = \psi(u^*au)$. Then*

$$pq\psi_u(a) = \langle \ell_\alpha(\omega), a^* \ell_{-\alpha}(\omega) \rangle_\phi, \quad a \in M.$$

Proof. We have from (2) and (4) that

$$\begin{aligned} \psi_u(a) &= \langle \Delta_{\psi,\phi}, u^* a^* u \rangle_\phi = [\Delta_{\psi,\phi} u^* a u]_\phi = [\Delta_{\psi,\phi}^{1/p} \Delta_{\psi,\phi}^{1/q} u^* a u]_\phi = \\ &= [u \Delta_{\psi,\phi}^{1/p}, \Delta_{\psi,\phi}^{1/q} u^* a]_\phi = \frac{1}{pq} \langle \ell_\alpha(\omega), a^* \ell_{-\alpha}(\omega) \rangle_\phi \end{aligned}$$

□

The L_p spaces for $1 < p < \infty$ are uniformly convex and uniformly smooth, therefore we can use the results of Section 2.

The map which sends the α -coordinate $x = \ell_\alpha(\omega)$ of ω onto the dual coordinate:

$$x \mapsto \tilde{x} := \ell_{-\alpha} \ell_\alpha^{-1}(x)$$

is called the duality map. It is easy to see that for $x \in L_p(M, \phi)$ we have

$$(7) \quad v_{x/\|x\|_p} = \left\| \frac{x}{p} \right\|_p^{1-p} \frac{\tilde{x}}{q}$$

and \tilde{x} is the unique element in $L_q(M, \phi)$, such that

$$(8) \quad \left\| \frac{\tilde{x}}{q} \right\|_q^q = \left\| \frac{x}{p} \right\|_p^p \quad \text{and} \quad \Re \langle x, \tilde{x} \rangle_\phi = pq \left\| \frac{x}{p} \right\|_p^p.$$

Proposition 5.2. *The duality map is a homeomorphism $L_p(M, \phi) \rightarrow L_q(M, \phi)$.*

Proof. Clearly, $pu\Delta_{\psi,\phi}^{1/p} \mapsto qu\Delta_{\psi,\phi}^{1/q}$ is continuous at 0. Further, let F be the map defined in Section 2 and $x \neq 0$, then we have from (7)

$$F(x) = \|x\|_p v_{x/\|x\|_p} = \frac{p^p}{pq} \|x\|_p^{2-p} \tilde{x}$$

The statement now follows from Theorem 2.2. □

Let us define the function $\Psi_p : L_p(M, \phi) \rightarrow R^+$ by

$$\Psi_p(x) = q \left\| \frac{x}{p} \right\|_p^p = q\varphi(1),$$

where $x = pu\Delta_{\varphi,\phi}^{1/p}$. Then we have

Proposition 5.3. Ψ_p is strongly differentiable. The strong derivative at x is given by

$$D_y \Psi_p(x) = \Re \langle y, \tilde{x} \rangle_\phi, \quad y \in L_p(M, \phi)$$

where \tilde{x} is the dual coordinate. If $1/p + 1/q = 1$, then

$$\Psi_q(\tilde{x}) = \Re \langle x, \tilde{x} \rangle_\phi - \Psi_p(x)$$

Proof. We have from the uniform smoothness of $L_p(M, \phi)$ that the norm is strongly differentiable at all points except $x = 0$ and

$$D_y \|x\|_p = \Re \langle y, v_{x/\|x\|_p} \rangle_\phi$$

It follows from (7) that for $x \neq 0$,

$$D_y \Psi_p(x) = q \left\| \frac{x}{p} \right\|_p^{p-1} \Re \langle y, v_{x/\|x\|_p} \rangle_\phi = \Re \langle y, \tilde{x} \rangle_\phi$$

As $p > 1$, the function $\left\| \frac{x}{p} \right\|_p^p$ is strongly differentiable at $x = 0$ and

$$D_y \Psi_p(0) = 0 = \Re \langle y, \tilde{0} \rangle_\phi$$

The last equality is rather obvious. □

In the commutative case, as well as on the manifold of positive definite $n \times n$ matrices, Ψ_p is the potential function in the sense of Amari, see [1] and [13, 10]. In general, it is not twice differentiable, but the above Proposition shows that the Legendre transformations, relating the dual coordinate systems, are still valid. It will be also clear from the results of the next Section, that

$$\Psi_q(\tilde{x}) = \sup_{y \in L_p(M, \phi)} (\Re \langle y, \tilde{x} \rangle_\phi - \Psi_p(y))$$

hence Ψ_q is the conjugate of the convex function Ψ_p .

6. DIVERGENCE IN $L_p(M, \phi)$.

Following [1], the function $D_p : L_p(M, \phi) \times L_p(M, \phi) \rightarrow R^+$, defined by

$$D_p(x, y) = \Psi_p(x) + \Psi_q(\tilde{y}) - \Re \langle x, \tilde{y} \rangle_\phi$$

is called the divergence. It has the following properties.

Proposition 6.1. (i) *Let $f_p(t) = p + qt^p - pqt$. Then*

$$(9) \quad D_p(x, y) \geq \left\| \frac{y}{p} \right\|_p^p f_p\left(\frac{\|x\|_p}{\|y\|_p}\right)$$

for all $x, y \in L_p(M, \phi)$, where for $y = 0$, we take the limit $\lim_{t \rightarrow 0} t^p f(s/t) = 0$ for all s . In particular, $D_p(x, y) \geq 0$ for all $x, y \in L_p(M, \phi)$ and equality is attained if and only if $x = y$.

(ii) *D_p is jointly continuous and strongly differentiable in the first variable.*

(iii) $D_p(y, x) = D_q(\tilde{x}, \tilde{y})$

(iv) $D_p(x, y) + D_p(y, z) = D_p(x, z) + \Re \langle x - y, \tilde{z} - \tilde{y} \rangle_\phi$

Proof. The statement (ii) follows from Proposition 5.3, (iii) and (iv) follow easily from the definition of D_p . We will now prove (i). If $y = 0$, then $D_p(x, y) = \Psi_p(x) \geq 0$. Similarly, if $x = 0$, $D_p(x, y) = \Psi_q(\tilde{y})$, which is equal to the right hand side of (9).

Let now $x \neq 0$, $y \neq 0$ and let $t = \|x\|_p / \|y\|_p$. Then by (7)

$$\Re \langle x, \tilde{y} \rangle_\phi = tq \left\| \frac{y}{p} \right\|_p^{p-1} \Re \left\langle \frac{x}{t}, v_{y/\|y\|_p} \right\rangle_\phi$$

Let $\|y\|_p = r$ and let S_r be the sphere with radius r in $L_p(M, \phi)$. Then $y, \frac{x}{t} \in S_r$. From Section 2, the tangent hyperplane $y + H$ to S_r at y is given by $\Re \langle z, v_{y/r} \rangle_\phi = r$, S_r lies entirely in the half-space given by $\Re \langle z, v_{y/r} \rangle_\phi \leq r$ and y is the unique point of S_r contained in $y + H$. Hence,

$$D_p(x, y) \geq \Psi_p(x) + \Psi_q(\tilde{y}) - tpq \left\| \frac{y}{p} \right\|_p^p = \left\| \frac{y}{p} \right\|_p^p f_p(t) \geq 0,$$

where equality is attained in the first inequality if and only if $\frac{x}{t} = y$, and in the second inequality if and only if $t = 1$. \square

We will also need the following lemma.

Lemma 6.1. *Let $y \in L_p(M, \phi)$, $d > 0$ and let*

$$U_{y,d} := \{x \in L_p(M, \phi), D_p(x, y) \leq d\}$$

Then $U_{y,d}$ is weakly closed, convex and contains no half-line.

Proof. It is easy to see that D_p is convex in the first variable, therefore the set $U_{y,d}$ is also convex. Next, let $\{x_\lambda\}$ be a net in $U_{y,d}$, converging weakly to some $x \in L_p(M, \phi)$ (it is in fact sufficient to consider sequences). Then $0 \leq D_p(x_\lambda, y) \leq d$ and we may suppose that the net $d_\lambda = D_p(x_\lambda, y)$ has a limit in $[0, d]$, using a subnet if necessary. We have

$$\lim_\lambda d_\lambda = \Psi_q(\tilde{y}) + \lim_\lambda \{q \|\frac{x_\lambda}{p}\|_p^p - \langle x_\lambda, \tilde{y} \rangle_\phi\}.$$

It follows that $\lim_\lambda \|x_\lambda\|_p$ exists. Furthermore, for u in the unit sphere of $L_q(M, \phi)$,

$$|\langle x, u \rangle_\phi| = \lim_\lambda |\langle x_\lambda, u \rangle_\phi| \leq \lim_\lambda \|x_\lambda\|_p$$

and hence $\|x\|_p \leq \lim_\lambda \|x_\lambda\|_p$. We therefore have

$$D_p(x, y) = \Psi_q(\tilde{y}) + q \|\frac{x}{p}\|_p^p - \langle x, \tilde{y} \rangle_\phi \leq \lim_\lambda d_\lambda \leq d$$

and $U_{y,d}$ is weakly closed.

Finally, let $h \neq 0$ and let $x_t = x + th$, $t \geq 0$ be a half-line in $L_p(M, \phi)$. For $y = 0$, we have $D_p(x_t, 0) = q \|\frac{x_t}{p}\|_p^p$. If $y \neq 0$, then by Proposition 6.1 (i),

$$D_p(x_t, y) \geq \|\frac{y}{p}\|_p^p f_p\left(\frac{\|x_t\|}{\|y\|}\right)$$

In both cases, the right-hand side goes to infinity as $t \rightarrow \infty$. Therefore $U_{y,d}$ can contain no half-line. □

7. D_p -PROJECTIONS.

Let C be a subset in $L_p(M, \phi)$, $y \in L_p(M, \phi)$. If there is a point $x_m \in C$, such that

$$D_p(x_m, y) = \min_{x \in C} D_p(x, y)$$

then x_m will be called a D_p -projection of y to C . In this section, we prove some uniqueness and existence results for D_p -projections.

Proposition 7.1. *Let C be a convex subset in $L_p(M, \phi)$, $y \in L_p(M, \phi)$ and $x_m \in C$. The following are equivalent.*

(i) $D_p(x_m, y) = \min_{x \in C} D_p(x, y)$

(ii) $\tilde{y} - \tilde{x}_m$ is in the normal cone to C at x_m , that is,

$$\Re \langle x - x_m, \tilde{y} - \tilde{x}_m \rangle_\phi \leq 0, \quad \forall x \in C$$

(iii) $D_p(x, y) \geq D_p(x, x_m) + D_p(x_m, y), \quad \forall x \in C$

If such a point exists, it is unique.

Proof. Let x_m be a point in C satisfying (i) and let $x \in C$. Then $x_t = tx + (1-t)x_m$ lies in C for all $t \in [0, 1]$ and thus $D_p(x_t, y) \geq D_p(x_m, y)$ on $[0, 1]$. We have from Proposition 5.3

$$0 \leq \frac{d}{dt^+} D_p(x_t, y)|_{t=0} = \Re \langle x - x_m, \tilde{x}_m - \tilde{y} \rangle_\phi$$

which is (ii). Further, from Proposition 6.1 (iv)

$$\Re \langle x - x_m, \tilde{x}_m - \tilde{y} \rangle_\phi = D_p(x, y) - D_p(x, x_m) - D_p(x_m, y),$$

hence (ii) implies (iii). Finally, let x_m satisfy (iii), then we clearly have $D_p(x_m, y) \leq D_p(x, y)$, for all $x \in C$.

To prove uniqueness, suppose that x_1 and x_2 are points in C , satisfying (iii). Then

$$D_p(x_1, y) \geq D_p(x_1, x_2) + D_p(x_2, y) \geq D_p(x_1, x_2) + D_p(x_2, x_1) + D_p(x_1, y).$$

It follows that $D_p(x_1, x_2) + D_p(x_2, x_1) \leq 0$ and hence $x_1 = x_2$. \square

Proposition 7.2. *Let C be a weakly compact subset in $L_p(M, \phi)$ and $y \in L_p(M, \phi)$. Then there exists a D_p -projection of y to C .*

Proof. For some $d > 0$, the set $U_{y,d}$ has a nonempty intersection with C . By Lemma 6.1, the sets $U_{y,d} \cap C$ are weakly closed. The intersection of these sets for all such d is therefore nonempty and is equal to some $U_{y,\rho} \cap C$. Then $\rho = \min_{x \in C} D_p(x, y)$ and all the points in $U_{y,\rho} \cap C$ are D_p -projections of y in C . \square

Proposition 7.3. *Let C be a weakly closed, convex, weakly locally compact subset in $L_p(M, \phi)$. Then for each $y \in L_p(M, \phi)$ there is a unique D_p -projection to C .*

Proof. Similarly as in the proof of previous Proposition, the set $U_{y,d} \cap C$ is non-empty for sufficiently large $d > 0$. By Lemma 6.1, this set is convex and weakly closed. As C is weakly locally compact, $U_{y,d} \cap C$ is also weakly locally compact. By [15], pp. 340, a closed convex locally compact subset in a locally convex space is compact if and only if it contains no half-line. It follows that $U_{y,d} \cap C$ are weakly compact and the intersection of all such nonempty sets is therefore nonempty. Each point in this intersection is a D_p -projection of y to C . By Proposition 7.1, such a point is unique. \square

Under the hypotheses of the above Proposition, we can define the map $y \mapsto x_m$, which sends each point y to its unique D_p -projection in C .

Proposition 7.4. *Let C be a weakly closed convex weakly locally compact subset in $L_p(M, \phi)$ and let $0 \in C$. Then the D_p -projection is continuous from $L_p(M, \phi)$ with its norm topology to C with the relative weak topology.*

Proof. Let $\{y^n\}$ be a sequence in $L_p(M, \phi)$ converging in norm to y . Let x_m^n be the unique D_p -projection of y^n and x_m be the unique D_p -projection of y in C from Proposition 7.3. We have to prove that x_m^n converges weakly to x_m .

Let $k > 0$ be such that $\|y^n\|_p \leq k$ for all n . Inserting $x = 0$ in Proposition (6.1), we get

$$0 \leq D_p(x_m, y) \leq \Psi_q(\tilde{y}) - \Psi_q(\tilde{x}_m)$$

and therefore by (8), $\|x_m\|_p \leq \|y\|_p \leq k$. Similarly, $\|x_m^n\|_p \leq \|y^n\|_p \leq k$ for each n .

As the duality map is continuous, we have $\tilde{y}^n \rightarrow \tilde{y}$ in $L_q(M, \phi)$. Further, we have from joint continuity of D_p that $\lim D_p(y, y^n) = \lim D_p(y^n, y) = D_p(y, y) = 0$. For sufficiently large n ,

$$\begin{aligned} d_n &:= D_p(x_m^n, y^n) = \inf_{x \in C, \|x\|_p \leq k} D_p(x, y^n) = \\ &= \inf_{x \in C, \|x\|_p \leq k} \{D_p(x, y) + D_p(y, y^n) - \Re \langle x - y, \tilde{y}^n - \tilde{y} \rangle_\phi\} \leq \\ &\leq D_p(x_m, y) + D_p(y, y^n) + 2k\|\tilde{y} - \tilde{y}^n\|_q \leq d + \varepsilon \end{aligned}$$

where $d := D_p(x_m, y)$. Further,

$$\begin{aligned} D_p(x_m^n, y) &= D_p(x_m^n, y^n) + D_p(y^n, y) - \Re \langle x_m^n - y^n, \tilde{y} - \tilde{y}^n \rangle_\phi \leq \\ &\leq d_n + D_p(y^n, y) + 2k\|\tilde{y} - \tilde{y}^n\|_q \leq d + 2\varepsilon \end{aligned}$$

Hence for sufficiently large n , $x_m^n \in U_{y, d+2\varepsilon} \cap C$. As in the proof of Proposition 7.3, these sets are nonempty weakly compact sets and therefore $\{x_m^n\}$ contains a weakly convergent subsequence. On the other hand, any limit of such subsequence has to be in $U_{y, d+2\varepsilon} \cap C$ for all ε and thus also in $\bigcap_\varepsilon U_{y, d+2\varepsilon} \cap C$. This intersection contains a single point x_m , it follows that x_m^n converges weakly to x_m . \square

8. THE α -DIVERGENCE IN M_*^+

Let $\alpha \in (-1, 1)$ and let $p = \frac{2}{1-\alpha}$. The divergence in $L_p(M, \phi)$, defines the functional $S_\alpha : M_* \times M_* \rightarrow R^+$, by

$$\begin{aligned} S_\alpha(\omega_1, \omega_2) &:= D_p(\ell_\alpha(\omega_1), \ell_\alpha(\omega_2)) = \\ &= q\varphi(1) + p\psi(1) - pq\Re \langle u\Delta_{\varphi, \phi}^{1/p}, v\Delta_{\psi, \phi}^{1/q} \rangle_\phi \end{aligned}$$

where $\omega_1(a) = \varphi(au)$ and $\omega_2(a) = \psi(av)$ are the polar decompositions. It is called the α -divergence. It follows from (6) that S_α does not depend from ϕ . In particular, if ψ is faithful, then

$$\langle u\Delta_{\varphi,\phi}^{1/p}, v\Delta_{\psi,\phi}^{1/q} \rangle_\phi = (\Delta_{\varphi,\xi_\psi}^{1/(2p)} \xi_\psi, \Delta_{\varphi,\xi_\psi}^{1/(2p)} u^* v \xi_\psi)$$

where ξ_ψ is a vector representative of ψ . It follows that if $\varphi, \psi \in M_*^+$, ψ is faithful and $\Delta_{\varphi,\psi} = \int \lambda E_\lambda$ is the spectral decomposition, then

$$S_\alpha(\varphi, \psi) = (g_p(\Delta_{\varphi,\psi}) \xi_\psi, \xi_\psi) = \int g_p(\lambda) \|E_\lambda \xi_\psi\|^2$$

where $g_p(t) = p + qt - pqt^{1/p}$. Hence, in this case the α -divergence is equal to the quasi entropy $S_{g_p}^1$, defined by Petz in [19, 18]. We will show that this is true on the whole of $M_*^+ \times M_*^+$.

Lemma 8.1. *Let $\varphi, \psi \in M_*^+$, $u, v \in M$ be partial isometries satisfying $u^*u = s(\varphi)$, $v^*v = s(\psi)$. Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$(10) \quad \langle u\Delta_{\varphi,\phi}^{1/p}, v\Delta_{\psi,\phi}^{1/q} \rangle_\phi = (\Delta_{\varphi,\xi_\psi}^{1/(2p)} \xi_\psi, \Delta_{\varphi,\xi_\psi}^{1/(2p)} u^* v \xi_\psi)$$

where ξ_ψ is a vector representative of ψ .

Proof. Let $1/p \leq 1/2$. We have

$$\langle u\Delta_{\varphi,\phi}^{1/p}, v\Delta_{\psi,\phi}^{1/q} \rangle_\phi = \lim_{y \rightarrow 1} (\Delta_{\psi,\phi}^{1/2-1/p} v^* u \Delta_{\varphi,\phi}^{1/p} \eta_\phi(y), \Delta_{\psi,\phi}^{1/2} \eta_\phi(y)),$$

with $y \in M_0$, $\|y\| \leq 1$. For $y \in N_\phi$,

$$(11) \quad \begin{aligned} & (\Delta_{\psi,\phi}^{1/2-1/p} v^* u \Delta_{\varphi,\phi}^{1/p} \eta_\phi(y), \Delta_{\psi,\phi}^{1/2} \eta_\phi(y)) = \\ & = (J_{\xi_\psi, \eta_\phi} \Delta_{\psi,\phi}^{1/2} \eta_\phi(y), J_{\xi_\psi, \eta_\phi} \Delta_{\psi,\phi}^{1/2-1/p} v^* u \Delta_{\varphi,\phi}^{1/p} \eta_\phi(y)) = \\ & = (y^* \xi_\psi, J_{\xi_\psi, \eta_\phi} \Delta_{\psi,\phi}^{1/2-1/p} v^* u \Delta_{\varphi,\phi}^{1/p} \eta_\phi(y)), \end{aligned}$$

here we have used that $J_{\xi_\psi, \eta_\phi}^* J_{\xi_\psi, \eta_\phi} = s(\psi) = s(\Delta_{\psi,\phi})$, the support of $\Delta_{\psi,\phi}$. Let $t \in \mathbb{R}$, then

$$\begin{aligned} J_{\xi_\psi, \eta_\phi} \Delta_{\psi,\phi}^{1/2-it} v^* u \Delta_{\varphi,\phi}^{it} \eta_\phi(y) &= S_{\xi_\psi, \eta_\phi} \Delta_{\psi,\phi}^{-it} v^* u \Delta_{\varphi,\phi}^{it} \eta_\phi(y) = \\ S_{\xi_\psi, \eta_\phi} v^* \Delta_{\psi,\phi}^{-it} \Delta_{\varphi_u, \phi}^{it} u \eta_\phi(y) &= S_{\xi_\psi, \eta_\phi} v^* (D\psi_v : D\varphi_u)_{-t} u \eta_\phi(y) = \\ &= y^* u^* (D\psi_v : D\varphi_u)_{-t}^* v \xi_\psi \end{aligned}$$

where $\varphi_u(a) = \varphi(u^*au)$ and $u\Delta_{\varphi,\phi}^{it} u^* = \Delta_{\varphi_u, \phi}^{it}$ by (C.8) in [4]. From this, we have

$$\begin{aligned} (y^* \xi_\psi, J_{\xi_\psi, \eta_\phi} \Delta_{\psi,\phi}^{1/2-it} v^* u \Delta_{\varphi,\phi}^{it} \eta_\phi(y)) &= ((D\psi_v : D\varphi_u)_{-t} u y y^* \xi_\psi, v \xi_\psi) = \\ &= (\Delta_{\psi_v, \xi_\psi}^{-it} \Delta_{\varphi_u, \xi_\psi}^{it} u y y^* \xi_\psi, v \xi_\psi) = (u \Delta_{\varphi, \xi_\psi}^{it} y y^* \xi_\psi, v \xi_\psi), \end{aligned}$$

where we have used (C.5) and (C.8) of [4]. It follows that for $z = it$,

$$(12) \quad (y^* \xi_\psi, J_{\xi_\psi, \eta_\phi} \Delta_{\psi, \phi}^{1/2-z} v^* u \Delta_{\varphi, \phi}^z \eta_\phi(y)) = (y^* \xi_\psi, y^* \Delta_{\varphi, \xi_\psi}^{\bar{z}} u^* v \xi_\psi).$$

By Lemma 3.1 in [17], both sides of (12) are holomorphic for $0 < \Re z < 1/2$ and continuous for $0 \leq \Re z \leq 1/2$. The equation (10) holds for $1/p \leq 1/2$ by (11) and analytic continuation of (12).

Let now $1/q \leq 1/2$. We have by the first part of the proof

$$\begin{aligned} \langle u \Delta_{\varphi, \phi}^{1/p}, v \Delta_{\psi, \phi}^{1/q} \rangle_\phi &= (u \xi_\varphi, v \Delta_{\xi_\psi, \xi_\varphi}^{1/q} \xi_\varphi) = (S_{\xi_\varphi, \xi_\psi} u^* v \xi_\psi, \Delta_{\xi_\psi, \xi_\varphi}^{1/q} S_{\xi_\varphi, \xi_\psi} \xi_\psi) \\ &= (J_{\xi_\psi, \xi_\varphi} \Delta_{\xi_\psi, \xi_\varphi}^{1/q} J_{\xi_\varphi, \xi_\psi} \Delta_{\xi_\varphi, \xi_\psi}^{1/2} \xi_\psi, \Delta_{\xi_\varphi, \xi_\psi}^{1/2} u^* v \xi_\psi) = \\ &= (\Delta_{\varphi, \xi_\psi}^{1/p-1/2} \xi_\psi, \Delta_{\varphi, \xi_\psi}^{1/2} u^* v \xi_\psi), \end{aligned}$$

we have used the equations (C.14) $J_{\eta_1, \eta_2}^* = J_{\eta_2, \eta_1}$ and $(\beta 5)$ $J_{\eta_1, \eta_2} \Delta_{\eta_1, \eta_2} J_{\eta_2, \eta_1} = \Delta_{\eta_2, \eta_1}^{-1}$ from Appendix C in [4]. \square

It follows that $S_\alpha(\varphi, \psi) = S_{g_p}^1(\varphi, \psi)$ for all positive normal functionals φ and ψ . The function g_p , $1 < p < \infty$ is operator convex and it follows from the results in [19] that

- (i) S_α is jointly convex on $M_*^+ \times M_*^+$
- (ii) S_α decreases under stochastic maps on $M_*^+ \times M_*^+$
- (iii) S_α is lower semicontinuous on $M_*^+ \times \mathcal{F}(M_*^+)$ endowed with the product of norm topologies, where $\mathcal{F}(M_*^+)$ denotes the set of faithful elements in M_*^+ .

The following properties of the α -divergence are valid on $M_* \times M_*$ and are immediate consequences of the results of Section 6.

- (i) Positivity

$$S_\alpha(\varphi, \psi) \geq \|\psi\|_1 g_p\left(\frac{\|\varphi\|_1}{\|\psi\|_1}\right) \geq 0$$

and $S_\alpha(\varphi, \psi) = 0$ if and only if $\varphi = \psi$ (here $\|\cdot\|_1$ is the norm in M_*).

- (ii) $S_\alpha(\varphi, \psi) = S_{-\alpha}(\psi, \varphi)$
- (iii) generalized Pythagorean relation

$$S_\alpha(\varphi, \psi) + S_\alpha(\psi, \sigma) = S_\alpha(\varphi, \sigma) + \Re \langle \ell_\alpha(\varphi) - \ell_\alpha(\psi), \ell_{-\alpha}(\sigma) - \ell_{-\alpha}(\psi) \rangle_\phi$$

Notice that the Pythagorean relation (iii) is a generalization of the classical version in [1], which says that equality is attained if and only if the α -geodesic connecting ψ and φ is orthogonal to the $-\alpha$ -geodesic connecting ψ and σ .

We also define the α -projection of $\varphi \in M_*$ onto a subset $C \subset M_*$ as the element in C that minimizes $S_\alpha(\cdot, \varphi)$ over C . We will say that a subset $C \subset M_*$ is α -convex if $\ell_\alpha(C)$ is convex. The next Proposition

is a generalization of the results in [1, 2] and follows directly from Proposition 7.1.

Proposition 8.1. *Let $C \subset M_*$ be α -convex and let $\psi \in M_*$, $\varphi_m \in C$. The following are equivalent.*

- (i) φ_m is an α -projection of ψ in C .
- (ii) For all $\sigma \in C$,

$$S_\alpha(\sigma, \psi) \geq S_\alpha(\varphi_m, \psi) + S_{-\alpha}(\varphi_m, \sigma)$$

- (iii) The curve $x_t \in L_q(M, \phi)$,

$$x_t := \ell_{-\alpha}(\varphi_m) + t(\ell_{-\alpha}(\psi) - \ell_{-\alpha}(\varphi_m))$$

lies in the normal cone to $\ell_\alpha(C)$ at $\ell_\alpha(\varphi_m)$ for all $t \geq 0$ (Note that $\ell_{-\alpha}^{-1}(x_t)$ is the $-\alpha$ -geodesic connecting φ_m and ψ .)

If such a point exists, it is unique.

The topology induced by the α -embedding from the norm, resp. the weak topology in $L_p(M, \phi)$ will be called the α -, resp. the α -weak topology. The following Proposition is also immediate from Section 7.

Proposition 8.2. *Let $C \subset M_*$ and let $\psi \in M_*$.*

- (i) *If C is α -weakly compact, then there exists an α -projection of ψ in C .*
- (ii) *If C is α -weakly closed, α -convex, α -weakly locally compact, then there exist a unique projection of ψ in C .*
- (iii) *If C is as in (ii) and, moreover, $0 \in C$, then the α -projection is a continuous map from M_* with the α -topology to C with the relative α -weak topology.*

Example 8.1. Let C be an extended α -family, generated by a finite number of positive elements, that is, there exist $x_1, \dots, x_n \in L_p(M, \phi)$, such that

$$\ell_\alpha(C) = \left\{ \sum_{i=1}^n t_i x_i, t_i \geq 0, i = 1, \dots, n \right\}$$

It follows from Proposition 8.2 (iii) that we have an α -projection from M_* to C , which is continuous in the α -topology.

9. THE CASE $\alpha = 0$.

Let $\alpha = 0$, $p = q = 2$. The space $L_2(M, \phi)$ can be identified with the Hilbert space H_ϕ and the dual pairing $\langle \cdot, \cdot \rangle_\phi$ is the inner product (\cdot, \cdot) in H_ϕ . Through this identification, the 0-embedding becomes the map

$$\omega \mapsto 2u\xi_\varphi$$

where $\omega(a) = \varphi(au)$ is the polar decomposition of ω and ξ_φ is the unique vector representative of φ in the neutral positive cone V in H_ϕ . Hence the 0-embedding maps M_* bijectively onto H_ϕ . In this case, the duality map is the identity on H_ϕ and the potential function is

$$\varphi_2(x) = \frac{1}{2}\|x\|^2$$

Therefore, the potential function is C^∞ -differentiable and

$$D_{y,z}^2\varphi_2(x) = \Re(y, z) \quad \forall x \in H_\phi$$

It follows that φ_2 defines a Riemannian metric in the tangent bundle $T\mathcal{M}_0$, which corresponds to the real part of the inner product, induced from the 0-embedding. In the matrix case, this metric was studied on density matrices and it was shown that it coincides with the Wigner–Yanase metric, see [9].

Up to multiplication by 2, the restriction of ℓ_0 to the positive cone M_*^+ corresponds to the identification of the positive normal functionals with elements in V proved by Araki in [3]. It has been also shown that this identification is a homeomorphism $M_*^+ \rightarrow V$. It follows that the relative 0-topology is the same as the relative L_1 -topology in M_*^+ .

The D_2 -divergence in H_ϕ is

$$D_2(x, y) = \frac{1}{2}\|x - y\|^2,$$

hence the D_2 -projection corresponds to minimizing the Hilbert space norm. This means, in particular, that there is a unique D_2 -projection onto every closed convex subset of H_ϕ .

The 0-divergence in M_* becomes

$$S_0(\omega_1, \omega_2) = 2\|u\xi_\varphi - v\xi_\psi\|^2$$

On the positive cone, the 0-divergence generalizes the classical Hellinger distance.

10. TOPOLOGIES INDUCED IN M_*^+

In this section, we study various topologies induced by the α -embeddings in M_*^+ . First of all, we see from Proposition 5.2 that the $+\alpha$ - and $-\alpha$ -topologies are the same. Let now $\varphi, \psi \in M_*^+$ and let $\ell_\alpha(\varphi) = x$, $\ell_\alpha(\psi) = y$. By Proposition 5.1 and (5), we have for $a \in M$,

$$\begin{aligned} |\varphi(a) - \psi(a)| &= |\langle x, a^*\tilde{x} \rangle_\phi - \langle y, a^*\tilde{y} \rangle_\phi| = \\ &= \frac{1}{2}|\langle (x+y), a^*(\tilde{x}-\tilde{y}) \rangle_\phi + \langle (x-y), a^*(\tilde{x}+\tilde{y}) \rangle_\phi| \leq \\ &\leq \frac{1}{2}\|a\|(\|x+y\|_p\|\tilde{x}-\tilde{y}\|_q + \|x-y\|_p\|\tilde{x}+\tilde{y}\|_q) \end{aligned}$$

It follows that the map $\ell_\alpha^{-1} : L_p^+(M, \phi) \rightarrow M_*^+$ is continuous relative to the norm topologies. Hence the α -topology is stronger than the L_1 -topology in M_*^+ .

Since the α -divergences can be seen as quasi-distances in M_*^+ , we will consider the topology induced by S_α , which will be called the S_α -topology. The S_α topology is given by the base of neighborhoods

$$O^\alpha(\psi, \varepsilon) := \{\varphi \in M_*^+, S_\alpha(\varphi, \psi) < \varepsilon\}$$

for $\psi \in M_*^+$, $\varepsilon > 0$. Because the functions $L_p(M, \phi) \ni x \mapsto D_p(x, y) \in \mathbb{R}^+$ are continuous for each y , the S_α -topology is weaker than the α -topology.

Lemma 10.1. *Let $\varphi, \psi \in M_*^+$ and let $-1 < \alpha \leq \beta < 1$. Then*

$$\begin{aligned} (1 - \beta)S_\beta(\varphi, \psi) &\leq (1 - \alpha)S_\alpha(\varphi, \psi) \\ (1 + \alpha)S_\alpha(\varphi, \psi) &\leq (1 + \beta)S_\beta(\varphi, \psi) \end{aligned}$$

Proof. The proof is essentially the same as in the classical case, see for example [16].

Let us consider the function

$$F_t(a) = t^a - at + a - 1 \quad a \in (0, 1)$$

Then F_t is convex on $(0, 1)$ for all $t \geq 0$. It follows that

$$\frac{F_t(1) - F_t(a)}{1 - a} \leq \frac{F_t(1) - F_t(b)}{1 - b}$$

for all $0 < a \leq b < 1$ and $t \geq 0$. As $F_t(1) = 0$ for all t , we get that the function $\frac{F_t(a)}{a-1}$ is increasing on $(0, 1)$. Let now $p = \frac{2}{1-\alpha}$ and put $a = 1/p$, then the function

$$\frac{F_t(1/p)}{1/p - 1} = 1/pg_p(t)$$

is decreasing on $(0, \infty)$. Hence we have for $0 < p \leq p' < \infty$

$$\frac{1}{p'}(g_{p'}(\Delta_{\varphi, \psi})\xi_\psi, \xi_\psi) \leq \frac{1}{p}(g_p(\Delta_{\varphi, \psi})\xi_\psi, \xi_\psi)$$

and the first inequality follows. The second inequality is obtained from the first and from $S_\alpha(\varphi, \psi) = S_{-\alpha}(\psi, \varphi)$. \square

From the last Lemma, we get for $\varphi \in M_*^+$, $-1 < \alpha \leq \beta < 1$ and $d > 0$,

$$O^\alpha(\psi, \frac{1 - \beta}{1 - \alpha}d) \subseteq O^\beta(\psi, d) \subseteq O^\alpha(\psi, \frac{1 + \alpha}{1 + \beta}d)$$

hence the S_α -topologies are the same for all $\alpha \in (-1, 1)$. In particular, these are the same as the S_0 -topology, which, by Section 9, is the same

as the 0-topology. It follows that on the positive cone, the topology induced from S_α coincides with the L_1 -topology .

11. THE UNIT SPHERE.

The α -embedding maps the unit sphere S in M_* onto the sphere S_p with radius p in $L_p(M, \phi)$. The duality map $x \mapsto \tilde{x}$ maps S_p onto the sphere S'_q with radius q in the dual space $L_q(M, \phi)$. From (7), we have that for $x \in S_p$,

$$(13) \quad \tilde{x} = qv_{x/p}$$

Proposition 11.1. *The duality map $S_p \ni x \mapsto \tilde{x} \in S'_q$ is uniformly continuous.*

Proof. The statement follows from (13) and Theorem 2.1. \square

Further, there is a unique tangent hyperplane $x + H_x$ through x , where H_x is given by the condition

$$\Re\langle y, \tilde{x} \rangle_\phi = q\Re\langle y, v_{x/p} \rangle_\phi = 0$$

Hence there is a splitting $L_p(M, \phi) = H_x \oplus [x]$ and, similarly as in [7], there is a continuous projection $\pi_x : L_p(M, \phi) \rightarrow H_x$, given by

$$\pi_x(y) = y - \Re\langle y, v_{x/\|x\|_p} \rangle_\phi \frac{x}{p} = y - \frac{1}{pq} \Re\langle y, \tilde{x} \rangle_\phi x,$$

which is obtained by minimizing the L_p -norm.

As the norm is strongly differentiable, the unit sphere can be given the structure of a differentiable submanifold \mathcal{D}_α in \mathcal{M}_α . If $\psi \in \mathcal{D}_\alpha$ has the α -coordinate $x \in S_p$, then the tangent space $T_x(\mathcal{D}_\alpha)$ can be identified with the tangent hyperplane H_x and π_x can be used to project the α -connection onto $T\mathcal{D}_\alpha$. But, even in the classical and the matrix case, the projected connection is no longer flat. Hence, it does not define a divergence, but nevertheless, we can use the restriction of S_α as a quasi-distance on S . This restriction has the form

$$S_\alpha(\omega_1, \omega_2) = pq(1 - \Re\langle u\Delta_{\varphi, \phi}^{1/p}, v\Delta_{\psi, \phi}^{1/q} \rangle_\phi)$$

which corresponds to the definition of the α -divergence in [1] for probability densities and in [12] for density matrices.

Let us now consider the topologies induced on the set of states $S^+ \subset M_*^+$. From [15] pp. 354, we have that the weak and the strong topologies coincide on the unit sphere of a uniformly convex space, hence these coincide on S_p . It follows that the relative α -topology and the α -weak topology are the same on S .

Let now $\varphi, \psi \in S$ and let $\ell_\alpha(\varphi) = x, \ell_\alpha(\psi) = y$. Then $x, y \in S_p \subset L_p(M, \phi)$ and

$$\left\| \frac{1}{2} \left(\frac{x}{p} + \frac{y}{p} \right) \right\|_p \geq \frac{1}{2pq} |\Re \langle x + y, \tilde{y} \rangle_\phi| = \left| 1 - \frac{1}{2pq} D_p(x, y) \right|$$

Therefore if $D_p(x, y) < 2pq\delta(\varepsilon)$, where $\delta(\varepsilon)$ is the module of convexity, then $\left\| \frac{1}{2} \left(\frac{x}{p} + \frac{y}{p} \right) \right\|_p > 1 - \delta(\varepsilon)$ and uniform convexity implies that $\|x - y\|_p < p\varepsilon$. It follows that for each $\varepsilon > 0$, the set $S_p \cap \ell_\alpha(O^\alpha(\psi, 2pq\delta(\varepsilon/p)))$ is contained in the strong neighborhood $S_p \cap \|x - y\|_p < \varepsilon$. Therefore, the S_α -topology coincides with the α -topology on S . We have proved the following

Proposition 11.2. *The topologies on S^+ , inherited from the α -topology, α -weak topology and S_α -topology coincide with the L_1 -topology for all $\alpha \in (-1, 1)$.*

Corollary 11.1. *The restriction of S_α to $S^+ \times S^+$ is continuous in the L_1 -topology.*

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